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Swiss Federal Institute of Technology Zurich

Covering Problems in Hypergraphs

Master Thesis

Michelle Sweering

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Advisors: M. Bucić, Prof B. Sudakov

Department of Mathematics, ETH Zürich

Abstract

An r-uniform hypergraph is a hypergraph in which each edge contains exactly r vertices. Such a hypergraph satisfies the (k,l)-covering property, if each k of its edges can be covered using l vertices.

In this thesis we study the transversal number of r-uniform hypergraphs satisfying the (k,l)-covering property. Specific regimes of interest include: small l, in particular $l = 2$, and large l, in particular $l = r$.

A special case of our covering property is directly related to the classical Ryser's conjecture about the relation between the transversal number and the matching number of r-uniform r-partite hypergraphs.

Contents

Chapter 1

Introduction

Many combinatorial problems can be formulated as finding the cover number of a certain hypergraph. Unfortunately, determining the cover number of hypergraphs is in many cases extremely difficult. In this thesis we prove bounds for r-uniform hypergraphs satisfying a certain natural property, but before we start we will first introduce the terminology used in this thesis.

1.1 Hypergraphs

Formally, a *hypergraph* is a pair of sets (V, E), where each element of E is a subset of V. The elements of V are called vertices, while the elements of E are called edges.

Note that usual graphs are precisely those hypergraphs in which each edge contains exactly two vertices. We call these 2-uniform hypergraphs. In general an *r-uniform hypergraph* is a hypergraph in which each edge contains exactly r verices.

Many properties of graphs can be extended to hypergraphs. The generalization of a bipartite graph is an *r-partite hypergraph*. This is a hypergraph whose vertex set can be partitioned into r vertex classes such that no edge contains more than one vertex from each vertex class.

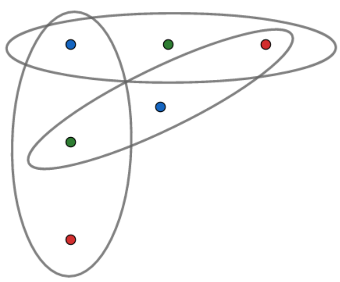

Figure 1.1: Tripartite 3-uniform hypergraph. Edges are depicted with ellipses and vertex classes with colours.

The *complete r-uniform hypergraph* on n vertices, denoted K_n^r, is the hypergraph consisting of n vertices and all possible edges of size r.

$$K_n^r = ([n], \{S \subseteq [n] : |S| = r\})$$

1.2 Covers

A transversal, or cover, of a hypergraph $H = (V, E)$ is a set of vertices $T \in V$ meeting every edge in E. The *transversal number* $\tau(H)$ of H is the size of the smallest such set. Note that this is an extension of the cover number of graphs.

The transversal number can also be viewed as the solution to the following discrete optimisation problem.

$$\begin{aligned}\tau(H) &= \min \sum_{v \in V} x_v \\ \text{s.t.} \quad & x_v \in \{0,1\} \quad \forall v \in V, \\ & \sum_{v \in E} x_v \geq 1 \quad \forall e \in E\end{aligned}$$

Here, x_v is the indicator variable of the transversal.

If we apply an LP-relaxation, i.e. don't require the x_v's to be integer, we obtain the so-called fractional cover number, which we will denote by $\tau^*(H)$.

$$\begin{aligned}\tau^*(H) &= \min \sum_{v \in V} x_v \\ \text{s.t.} \quad & 0 \leq x_v \leq 1 \quad \forall v \in V, \\ & \sum_{v \in E} x_v \geq 1 \quad \forall e \in E\end{aligned}$$

In this case multiple vertices can jointly "cover" an edge when their combined weight is at least 1. Observe that the fractional cover number can never be greater than the transversal number. However it can sometimes be smaller as shown in the example below.

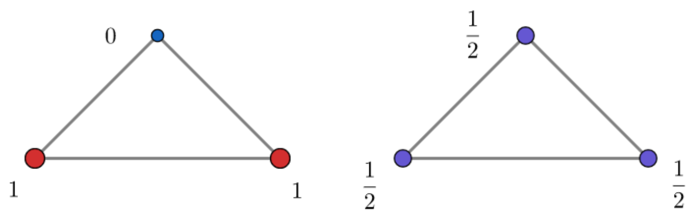

Figure 1.2: Graph with cover number 2 and fractional cover number 1.5

Fractional cover numbers are sometimes easier to work with. For example, fractional cover numbers are computable in polynomial time, while finding the transversal number is NP-hard. This follows from the NP-completeness of determining the vertex cover number, see Karp [7].

1.3 Matchings

Another property which can be extended to hypergraphs is the *matching number*. Just like in the 2-uniform case, a matching is a set of disjoint edges and the matching number maximal size of such a set.

$$\begin{aligned} \nu(H) &= \max \sum_{e \in E} y_e \\ \text{s.t.} \quad & y_e \in \{0,1\} \quad \forall e \in E, \\ & \sum_{v \in e} y_e \leq 1 \quad \forall v \in V \end{aligned}$$

Here, y_e is the indicator variable of the matching. This is again an integer linear program and one may consider an LP-relaxation giving rise to the following fractional matching problem:

$$\begin{aligned} \nu^*(H) &= \max \sum_{e \in E} y_e \\ \text{s.t.} \quad & 0 \leq y_e \leq 1 \quad \forall e \in E, \\ & \sum_{v \in e} y_e \leq 1 \quad \forall v \in V \end{aligned}$$

Note that $x_v \leq 1$ for all $v \in V$ and $y_e \leq 1$ for all $e \in E$ are redundant constraints and omitting them gives two dual linear programs. By strong duality,

$$\nu(H) \leq \nu^*(H) = \tau^*(H) \leq \tau(H).$$

Transversals and matchings of hypergraphs are closely related. For example, the union of any maximal matching is a transversal.

1.4 (k,l)-Covering property

We say a hypergraph H satisfies the *(k,l)-covering property*, or for short that H is a (k,l)-hypergraph, when any k edges of H can be covered using l vertices. The main problem of this thesis is to determine the maximum transversal number an r-uniform (k,l)-hypergraph can have, for a variety of regimes of r, k and l.

$$h_r(k,l) = \max_{H \in r\text{-uniform } (k,l)\text{-hypergraphs}} \tau(H)$$

If no maximum exist, we say $h_r(k,l) = \infty$.

The problem was introduced by Erdős et al. [6] and recently used to solve problems about partitioning random graphs into monochromatic components by Bucić et al. [4].

A variation on this problem additionally requires the hypergraph to be r-partite. In that case we denote the maximum transversal number with $h_r^r(k,l)$.

1.5 Outline

We will now give a brief overview of the rest of this thesis.

In chapter 2 we will give an overview of some important results in the area of hypergraph coverings from Kőnig's theorem and Ryser's conjecture up recent work.

In chapter 3 we bound the size of complete r-uniform (k,l)-hypergraphs: the lower bound using a greedy algorithm and the upper bound using a construction.

In chapter 4 we will calculate $h_r(k,l)$ explicitly for the values it is known: $l = 1$, $l = 2 \wedge k \leq 6$, $k = l + 1$ and $k = l + 2$.

In chapter 5 we construct a new hypergraph, with the the edges of the original hypergraph as its vertices and show how this can be used to find another bound for complete graphs.

In chapter 6 we will calculate and compare upper and lower bounds on $h_r(k,l)$ for the special case $r = l$.

In chapter 7 we will discuss how this all relates to Ryser's conjecture, a famous open problem.

Chapter 2

Previous Results

In this chapter we will briefly describe various key papers and results in the area.

2.1 Kőnig's theorem '31

One of the first important results in the history of coverings is Kőnig's theorem [13], also known as the Kőnig-Egerváry theorem and sometimes spelled with an umlaut.

Theorem 2.1 (Kőnig) *If a graph is bipartite, then its matching number equals its cover number.*

The proof from Kőnig's book used alternating paths. However, we will give an alternative proof using the max-flow-min-cut theorem.

Proof Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with vertex classes V_1 and V_2. We construct a new graph G' by adding two vertices s and t, joining s with all vertices in A and joining t with all vertices in B. Observe that the max-flow from s to t equals the matching number of G, while the min-cut equals the cover number of G. By the max-flow-min-cut theorem these two quantities are equal. $\square$

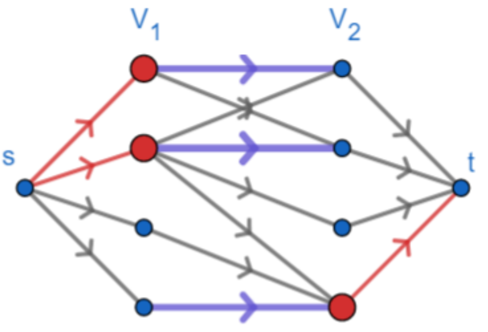

Figure 2.1: Bipartite graph with minimum cut/cover (red) and maximum flow/matching (purple)

2.2 Ryser's conjecture '71

A generalisation of Kőnig's theorem to hypergraphs is Ryser's conjecture, which describes the relationship between the transversal number τ and matching number ν of r-uniform r-partite hypergraphs.

Conjecture 2.2 *Let H be an r-uniform r-partite hypergraph. Then*

$$\tau(H) \leq (r-1)\nu(H).$$

This conjecture is often attributed to a paper of Ryser [15], but according to Best and Wanless [3] it was first published in the PhD thesis of Henderson [12].

The case of 3-uniform tripartite hypergraph has attracted a lot of attention over the years. The trivial bound of 3ν was improved to $\frac{25}{9}\nu$ by Szemerédi and Tuza in 1982 [17], then to $\frac{8}{3}\nu$ by Tuza in 1987 [18], to $\frac{5}{2}\nu$ by Haxell in 1995 [11] and finally to the optimal bound 2ν by Aharoni in 2001 [1].

We will discuss Ryser's conjecture, including Aharoni's proof for $r = 3$, in chapter 7.

2.3 Lovász' integrality gap '75

Another big result in the area comes from Lovász [14]. He proves the ratio between the transversal number and the fractional cover number does not exceed $1 + \ln d$, where d is the maximum degree of the graph. The fractional cover number is easier to compute, because it is an LP, as seen in section 1.2. We will now explain the proof from the paper.

Theorem 2.3 (Lovász) *Let H be a hypergraph of maximum degree d, transversal number τ and fractional cover number τ^*. Then $\tau \leq (1 + \ln d)\tau^*$.*

Proof The proof makes use of the "greedy cover algorithm". This algorithm constructs a transversal by starting with the empty set and repeatedly adding the vertex which is in most uncovered edges until all edges are covered.

The first vertex covers exactly d edges. Thereafter, the number of new edges each new vertex covers is non-increasing. We define t_i to be the number of vertices of the transversal which cover i new edges. Note that

$$\tau \leq t_1 + \cdots + t_d \quad \text{and} \quad |E| = t_1 + 2t_2 + \cdots + dt_d.$$

The fractional cover number on the other hand satisfies $\tau^* \geq |E|/d$, since the maximum degree is d and thus every vertex can cover at most d edges.

Let E_i be the set of uncovered edges after $t_{i+1} + \cdots t_d$ steps. Note that the hypergraph spanned by those edges has maximum degree i giving us the equations

$$|E_i| = t_1 + 2t_2 + \cdots + it_i \quad \text{and} \quad \tau^* \geq |E_i|/i.$$

Taking appropriate linear combinations we find

$$\begin{aligned}\tau \leq t_1 + \cdots + t_d &= \frac{t_1 + \cdots + dt_d}{d+1} + \sum_{i=1}^{d} \frac{t_1 + \cdots + it_i}{i(i+1)} \\ &\leq \frac{d\tau^*}{d+1} + \sum_{i=1}^{d} \frac{\tau^*}{i+1} \leq (1 + \ln d)\tau^*. \quad \square\end{aligned}$$

2.4 Fractional Ryser's conjecture '88

A weaker version of Ryser's conjecture, which bounds the fractional cover number, was proven by Füredi [10] and earlier by Gyárfás (cited in [9]).

Theorem 2.4 *Let H be an r-uniform r-partite hypergraph. Then*

$$\tau^*(H) \leq (r-1)\nu(H).$$

2.5 Erdős, Hajnal and Tuza's problem '91

The main focus of this thesis is the problem introduced by Erdős, Hajnal and Tuza in [6].

Question 2.5 *Let H be an r-uniform hypergraph such that any k edges of H can be covered using l vertices. What is the maximal transversal number H can have?*

Several values were found for small l, which we will discuss in chapter 4.

In 2019 the parameter $h_r(k,l)$ was rediscovered by Bucić, Korándi and Sudakov [4] in the context of covering edge coloured random graphs with monochromatic trees. In this context $l = r$ is the relevant case, which we will discuss in chapter 6.

Chapter 3

Complete Graphs

Recall that a hypergraph G satisfies the (k,l)-covering property, if every k edges can be covered by l vertices. Let K_n^r be the complete r-uniform hypergraph on n vertices. In this chapter we will prove bounds on the size of complete uniform (k,l)-hypergraphs.

Note that property (k,l) is trivially satisfied when $\binom{n}{r} < k$ (because the graph has less than k edges) or $k \leq l$ (since we can pick a vertex for each edge).

3.1 Small cases $l = 1$ or $r = 1$

We will discuss these cases separately, because they can be determined exactly and addressing them here will simplify our proofs later on.

Proposition 3.1 *Let $k, l, n, r \in \mathbb{N}$ with $\binom{n}{r} \geq k > l \geq 1$.*

(a) K_n^r satisfies $(k,1)$ if and only if $n < \frac{kr}{k-1}$.

(b) K_n^1 never satisfies (k,l).

Proof

(a) Suppose $n < \frac{kr}{k-1}$. Then $(n-r)k < n$. For any k edges $e_1, ..., e_k$, we have

$$|e_1 \cap ... \cap e_k| = n - |e_1^C \cup ... \cup e_k^C| \geq n - k(n-r) > 0$$

So there exists a vertex which covers all k edges and K_n^r satisfies $(k,1)$.

Suppose $n \geq \frac{kr}{k-1}$. Then $(n-r)k \geq n$. We pick k edges whose complements cover all n vertices: $e_i = \{i(n-r)+1, ..., (i+1)(n-r)\}^C$ (vertices mod n). Every vertex is in the complement of some edge, so we cannot cover all edges with a single vertex. Therefore K_n^r does not satisfy $(k,1)$.

(b) We can never cover k distinct edges of size 1 (i.e. vertices) using $l < k$ vertices. □

3.2 Lower bound

Theorem 3.2 *Let $k, l, n, r \in \mathbb{N}$ with $n < \frac{rl}{\ln k}$. Then K_n^r does satisfy property (k, l).*

Proof Consider k edges of K_n^r. We will use the greedy algorithm where we keep picking the vertex which covers most uncovered edges. Let a_i be the number of uncovered edges after selecting i vertices. Note that

$$a_0 = k \text{ and } a_{i+1} \leq a_i - \left\lceil \frac{a_i \cdot r}{n-i} \right\rceil = \left\lfloor a_i \cdot \frac{n-i-r}{n-i} \right\rfloor \leq a_i \cdot \frac{n-r}{n}.$$

By induction on m

$$a_m \leq \left\lfloor k \cdot \left(1 - \frac{r}{n}\right)^m \right\rfloor \leq \lfloor k e^{-rm/n} \rfloor.$$

Substituting $n < \frac{rl}{\ln k}$ gives $a_l = 0$. Therefore l vertices suffice to cover all k edges. $\square$

3.3 Upper bound lemma

A graph which doesn't satisfy property (k, l), contains a subhypergraph of k edges with transversal number greater than l. In the following lemma we construct such subhypergraphs with high transversal number.

Lemma 3.3 *Let $n, r, k, l, a, b \in \mathbb{N}$ such that $r \geq a$, $n \geq \lceil \frac{r}{a} \rceil \cdot b$, $k \geq \binom{b}{a}$ and $l \leq b - a$. Then K_n^r does not satisfy property (k, l).*

Proof First, we partition the vertices of K_n^r into b sets of at least $\lceil \frac{r}{a} \rceil$ vertices each. We then pick an edge in the union of each a of these sets such that it is not in the union of any $a - 1$ sets, obtaining $\binom{b}{a}$ edges in total. These edges cannot be covered by $b - a$ vertices. Therefore K_n^r does not satisfy property $(\binom{b}{a}, b - a)$. $\square$

Picking suitable values of a and b gives us an upper bound on n of approximately $\frac{rl}{\log_l k} + r$ for $k \leq l^r$ and $r + l$ for $k > l^r$.

3.4 Upper bound for relatively small k

Theorem 3.4 *Let $k, l, n, r \in \mathbb{N}$ with $\binom{n}{r} \geq k > l \geq 2$, $r \geq 2$, $l^r \geq k \geq l^4$ and*

$$n \geq \left\lceil \frac{r}{\lfloor \log_l k \rfloor} \right\rceil (l + \lfloor \log_l k \rfloor).$$

Then K_n^r does not satisfy property (k, l).

Proof We prove this case using lemma 3.3, setting $a := \lfloor \log_l k \rfloor$ and $b := l + a$. It remains to be proven that $k \geq \binom{b}{a}$, or equivalently $k \geq \binom{b}{l}$.

We will first check the small cases. Note that $a \geq 4$.

- If $l = 2$, then $2^a \leq k$ and $\binom{b}{a} = \binom{a+2}{a} = \frac{(a+2)(a+1)}{2}$. Therefore $\binom{b}{a} \leq k$.
- If $l = 3$, then $3^a \leq k$ and $\binom{b}{a} = \binom{a+3}{a} = \frac{(a+3)(a+2)(a+1)}{6}$. Therefore $\binom{b}{a} \leq k$.
- If $l = 4$, then $4^a \leq k$ and $\binom{b}{a} = \binom{a+4}{a} = \frac{(a+4)(a+3)(a+2)(a+1)}{24}$. Therefore $\binom{b}{a} \leq k$.

Now suppose $l \geq 5$. Substituting a and b into the binomial coefficient gives

$$\binom{b}{a} \leq \left(\frac{b}{\sqrt{a}}\right)^a = \left(\frac{b}{l\sqrt{a}}\right)^a \cdot l^a \leq \left(\frac{l+a}{l\sqrt{a}}\right)^a \cdot k.$$

It follows that $\binom{b}{a} \leq k$ whenever $l + a \leq l\sqrt{a}$, which holds if and only if

$$\frac{l(l-2-\sqrt{l^2-4l})}{2} \leq a \leq \frac{l(l-2+\sqrt{l^2-4l})}{2}.$$

The first inequality holds in our case, since

$$\frac{l(\sqrt{l^2-4l+4}-\sqrt{l^2-4l})}{2} \leq \frac{l}{\sqrt{l^2-4l}} < 3 \leq a.$$

If the second inequality doesn't hold, then $a > l(l-3)$. In that case the above bound on $\binom{b}{a}$ isn't very sharp, since a is much larger than l. Using different inequalities we get

$$\binom{b}{a} = \binom{b}{l} \leq \left(\frac{b}{\sqrt{l}}\right)^l = \frac{b^l}{l^{a+l/2}} \cdot l^a \leq \frac{(a+l)^l}{l^{a+l/2}} \cdot k.$$

It follows that $\binom{b}{a} \leq k$ whenever $a + l \leq l^{a/l+1/2}$. Substituting $a = l(l-3)$ gives $l(l-2) \leq l^{l-2.5}$, which holds for all $l \geq 4$. The inequality also holds for all higher values of a, since the right-hand side grows faster.

$$\frac{d}{da}\, a + l = 1 \leq \frac{\ln l}{l} l^{a/l+1/2} = \frac{d}{da}\, l^{a/l+1/2}$$

Here ln denotes the natural logarithm. We conclude that K_n^r does not satisfy property (k, l). □

Remark 3.5 *The bound on n also holds when $l^2 > k \geq l$, because then K_n^r contains $l+1$ independent edges.*

Remark 3.6 *When $l^4 > k \geq l^2$, the bound on n holds for $l \geq 4$.*

Remark 3.7 *This bound is not tight in general, see chapter 6.*

3.5 Exact answer large k

For $k > l^r$ we can determine the answer exactly.

Theorem 3.8 *Let $k, l, n, r \in \mathbb{N}$ with $\binom{n}{r} \geq k > l \geq 2$, $r \geq 2$ and $k > l^r$. Then K_n^r satisfies (k, l) if and only if*

$$n < r + l \text{ or } (k, l, n, r) \in \{(5, 2, 4, 2), (9, 2, 5, 3)\}.$$

Proof Note that the theorem holds, when $r = l = 2$.

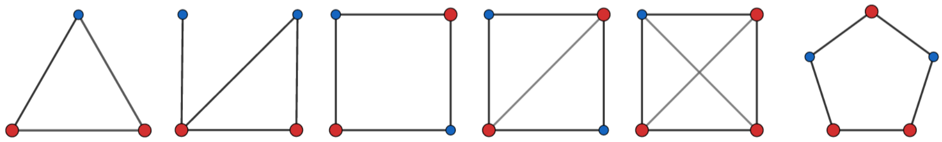

Figure 3.1: Edge cases

If $n < r + l$ any l vertices cover all edges.

Otherwise $n \geq r + l$ and we apply lemma 3.3 on $a = r$ and $b = r + l$.

If $l = 2$ and $r = 3$, then $\binom{r+l}{r} = 10$. So if $k \geq 10$, then K_n does not satisfy $(k, 2)$ by lemma 3.3. We also know that $k > l^r = 8$. The only remaining case is $k = 9$, $n \geq 5$. If $n = 5$, we can cover any nine edges with two vertices by picking the vertices in the complement of the only absent edge. If $n \geq 6$, this isn't possible, because the hypergraph

$$(\{a, b, c, d, e, f\}, \{\{a, b, f\}, \{b, c, d\}, \{a, c, e\}, \{a, d, e\}, \{b, e, f\}, \{c, d, f\}\})$$

has transversal number 3.

If $l = 2$ and $r \geq 4$, then $\binom{r+l}{r} = \frac{(r+1)(r+2)}{2} \leq r^2 \leq 2^r < k$.

If $l = 3$, then $\binom{r+l}{r} = \frac{(r+1)(r+2)(r+3)}{6} \leq 3^r + 1 \leq k$.

If $l \geq 4$, then $\binom{r+l}{r} = \frac{(l+1)(l+2)}{2} \cdot \prod_{i=3}^{r} \frac{l+i}{i} \leq l^r < k$.

In these last three cases K_n^r does not satisfy property (k, l), again by lemma 3.3.

3.6 Connection to codes

We can identify the edges of the hypergraphs on n vertices with codewords of weight r in $\{0, 1\}^n$. This gives us constant weight codes or r-out-of-n-codes. We conjecture that hypergraphs corresponding to efficient constant weight codes tend to have a high cover number. Constructing such a code may bring the upper bound closer to the lower bound.

Chapter 4

Exact Transversal Numbers

In the next few chapters we will discuss the following problem. Let H be an r-uniform hypergraph. If any k edges have a transversal of size l, what is the maximal size $h_r(k,l)$ the transversal of H can have?

Calculating $h_r(k,l)$ turns out to be difficult. In this chapter we will calculate $h_r(k,l)$ exactly for $k = l+1$, $k = l+2$ and several small values of l and k for which the exact value has been determined. In later chapters we will study the asymptotic behaviour for more general regimes.

$k \backslash l$	1	2	3	4	5	6	7
1	∞	∞	∞	∞	∞	∞	∞
2	r	∞	∞	∞	∞	∞	∞
3	$\lceil r/2 \rceil$	$2r$	∞	∞	∞	∞	∞
4	$\lceil r/3 \rceil$	$\lceil 3r/2 \rceil$	$3r$	∞	∞	∞	∞
5	$\lceil r/4 \rceil$	$\lceil 5r/4 \rceil$	$\lceil 5r/2 \rceil$	$4r$	∞	∞	∞
6	$\lceil r/5 \rceil$	r	?	$\lceil 7r/2 \rceil$	$5r$	∞	∞
7	$\lceil r/6 \rceil$	?	?	?	$\lceil 9r/2 \rceil$	$6r$	∞
8	$\lceil r/7 \rceil$	?	?	?	?	$\lceil 11r/2 \rceil$	$7r$

Table 4.1: $h_r(k,l)$ for small k and l

Note that all hypergraphs automatically satisfy (k,l) for $k \leq l$. Therefore the transversal number of such r-uniform (k,l)-hypergraphs is unbounded.

If $k \geq l+1$ however, then the matching number is at most l and the transversal number is at most rl. Thus $h_r(k,l)$ is finite.

4.1 Case $l = 1$

In this section we will calculate the transversal number of r-uniform graphs in which every k edges have a vertex in common. This is a well known result, found in Füredi [10] for example.

Theorem 4.1 *For all $r, k \in \mathbb{N}$ with $k \geq 2$ it holds that*

$$h_r(k,1) = \left\lfloor \frac{r-1}{k-1} \right\rfloor + 1.$$

Proof We will construct an example for the lower bound and prove the upper bound by induction.

4.1.1 Lower bound

An example of an r-uniform $(k,1)$-hypergraph with transversal number $\lfloor \frac{r-1}{k-1} \rfloor + 1$ is the complete r-hypergraph on $r + \lfloor \frac{r-1}{k-1} \rfloor$ vertices. Every k edges have at least

$$r + \left\lfloor \frac{r-1}{k-1} \right\rfloor - k \cdot \left\lfloor \frac{r-1}{k-1} \right\rfloor \geq 1$$

vertex in common, so it satisfies property $(k,1)$. Moreover, it has transversal number $\lfloor \frac{r-1}{k-1} \rfloor + 1$.

4.1.2 Upper bound

We will prove the upper bound by induction.

Induction Basis If any two edges intersect, any edge will be a transversal. Therefore $h_r(2,1) \leq r = \lfloor \frac{r-1}{2-1} \rfloor + 1$.

Induction Hypothesis For some $k \geq 3$ it holds that if any $k-1$ edges mutually intersect, then the transversal number is at most $\lfloor \frac{r-1}{k-2} \rfloor + 1$

Induction Step Let $H = (V, E)$ be an r-uniform hypergraph such that any $k \geq 3$ edges mutually intersect. Pick one edge e.

The edges which contain at least $r - \lfloor \frac{r-1}{k-1} \rfloor$ vertices of e, can be covered using by any $\lfloor \frac{r-1}{k-1} \rfloor + 1$ vertices of e.

The other edges of $E \backslash e$ restricted to e have size at most $r - \lfloor \frac{r-1}{k-1} \rfloor - 1$ and satisfy the $(k-1,1)$ property. By the induction hypothesis we can cover these edges using

$$\left\lfloor \frac{r - \lfloor \frac{r-1}{k-1} \rfloor - 2}{k-2} \right\rfloor + 1 \leq \left\lfloor \frac{r-1}{k-1} \right\rfloor + 1$$

vertices.

We conclude that $h_r(k,1) = \lceil \frac{r-1}{k-1} \rceil + 1$ for all $k \geq 2$. $\square$

4.2 Case $k = l + 1$

In this section we will prove the formula of Erdős, Fon-Der-Flaass, Kostochka and Tuza [5] for $h_r(l+1,l)$.

Proposition 4.2 *For all $r, l \in \mathbb{N}$ we have*

$$h_r(l+1,l) = rl.$$

Proof Any $l+1$ edges can be covered using at most l vertices. So any matching consists of at most l edges and rl vertices. The vertices of any maximal matching form a transversal. Therefore $h_r(l+1,l) \leq rl$.

The other direction holds, since the complete r-uniform hypergraph on $r(l+1)-1$ vertices, satisfies property $(l+1,l)$ and has transversal number rl. □

4.3 Case $k = l + 2$

In this section we will discuss the case $k = l+2$. Like the the other diagonal case, there is a nice formula in [5], which is relatively easy to prove.

Proposition 4.3 *For all $r, l \in \mathbb{N}$ we have*

$$h_r(l+2,l) = \left\lceil \frac{(2l-1)r}{2} \right\rceil.$$

For the upper bound we will use the following lemma.

Lemma 4.4 *For all $r, l, k \in \mathbb{N}$ with $k > l$ we have*

$$h_r(k+1,l+1) \leq h_r(k,l) + r.$$

Proof Let H be an r-uniform $(k+1,l+1)$-hypergraph with transversal number $h_r(k+1,l+1)$. Pick an edge e in H and define H' to be H with all vertices of e removed. Note that H' satisfies property (k,l). It follows that

$$h_r(k+1,l+1) = \tau(H) \leq \tau(H') + r \leq h_r(k,l) + r.$$ □

Proof (Proposition 4.3) Combining lemma 4.4 with $h_r(3,1) = \lceil r/2 \rceil$ from theorem 4.1 and using induction on l, we get

$$h_r(l+2,l) \leq \left\lceil \frac{(2l-1)r}{2} \right\rceil.$$

The other direction holds, because the disjoint union of two complete r-uniform hypergraph on $rl-1$ and $r + \lceil r/2 \rceil - 1$ vertices respectively satisfies property $(l+2,l)$ and has transversal number

$$r(l-1) + \lceil r/2 \rceil = \left\lceil \frac{(2l-1)r}{2} \right\rceil.$$ □

4.4 Case $l = 2$

Erdős, Fon-Der-Flaass, Kostochka and Tuza proved $h_r(k, 2)$ for k up to 6. In these cases the complete graph gives the optimal lower bound, while this is no longer the case for $k = 7$, see Fon-Der-Flaass [8]. We have shown in sections 4.2 and 4.3 that $h_r(3, 2) = 2r$ and $h_r(4, 2) = \lceil 3r/2 \rceil$ respectively. In this section we give proofs for $k = 5, 6$.

4.4.1 Case $k = 5$

Proposition 4.5 *For all $r \in \mathbb{N}$ it holds that*

$$h_r(5, 2) = \lceil 5r/4 \rceil.$$

Proof For the lower bound we consider K_n^r, the complete r-uniform hypergraph on $n = r + \lceil 5r/4 \rceil - 1$ vertices. Its transversal number is $\lceil 5r/4 \rceil$.

Consider five of its edges e_1, e_2, e_3, e_4, e_5. We will prove by contradiction that they can be covered using 2 vertices.

If four of them mutually intersect, then we can cover all five using a vertex from the intersection and a vertex from the remaining edge. This is a contradiction, so there are at least

$$5r - 2(r + \lceil 5r/4 \rceil - 1) = r - 2\lceil r/4 \rceil + 2$$

vertices with degree 3. If three edges mutually intersect the remaining two edges must be disjoint; otherwise we pick one vertex in the intersection of the three edges and one in the intersection of the remaining two edges. It follows that there are at least three pairs of mutually intersecting edges, because $r - 2\lceil r/4 \rceil + 2 - 2(\lceil r/4 \rceil - 1) > 0$.

If one edge is disjoint to all others, then there exists a vertex of degree at least $4r/(\lceil 5r/4 \rceil - 1) > 3$. This is contradiction again, so all edges intersect at least one other edge.

It follows that without loss of generality $e_1 \cap e_2 \cap e_3$, $e_1 \cap e_2 \cap e_4$ and $e_1 \cap e_2 \cap e_5$ are non-empty. In that case e_3, e_4 and e_5 are disjoint, which is a contradiction, because $r + \lceil 5r/4 \rceil - 1 < 3r$.

For the upper bound, let $H = (V, E)$ be an r-uniform $(5, 2)$-hypergraph. Let $e_1 \in E$. If all edges intersect e_1, then e_1 is a transversal. Otherwise there exists an edge e_2 disjoint from e_1.

Suppose $|e_2 \cap e_3| > \lfloor 3r/4 \rfloor$ for all e_3 disjoint to e_1. Then the union of e_1 and any $\lceil r/4 \rceil$ vertices of e_2 is a transversal, so $\tau(H) \leq r + \lceil r/4 \rceil = \lceil 5r/4 \rceil$.

Suppose $|e_2 \cap e_3| \leq \lfloor 3r/4 \rfloor$. Any two edges disjoint to $e_2 \cap e_3$ intersect on e_1. We can cover those edges using $\lceil r/2 \rceil$ vertices (cf. proof 4.1.2). Therefore $\tau(H) \leq |e_2 \cap e_3| + \lceil r/2 \rceil \leq \lceil 5r/4 \rceil$. □

4.4.2 Case $k = 6$

Proposition 4.6 *For all $r \in \mathbb{N}$ it holds that*

$$h_r(6, 2) = r.$$

Proof For the lower bound we consider K^r_{2r-1}, the complete r-uniform hypergraph on $2r - 1$ vertices. Its transversal number is r. To show that it satisfies the property $(6, 2)$, we consider the hypergraph H spanned by six of its edges. The average degree of H is $\frac{6r}{2r-1} > 3$, so there is a vertex with degree at least four. The two (or less) edges which it doesn't cover, intersect. Therefore K^r_{2r-1} satisfies $(6, 2)$ and $h_r(6, 2) \geq r$.

For the upper bound, we prove by contradiction that $h_r(6, 2) \leq r$. Suppose H is an r-uniform $(6, 2)$-hypergraph with transversal number greater than r.

If there exists an edge which intersects all other edges, it is a transversal. Therefore each edge has an edge disjoint to it. We pick a pair $\{e_1, e_2\}$ of disjoint edges.

Suppose there exists an edge e_3 such that $|e_1 \cap e_3| \leq r/2$ and $|e_2 \cap e_3| \leq r/2$. Let $A \supseteq e_1 \cap e_3$ and $B \supseteq e_2 \cap e_3$ such that $|A| = |B| \leq r/2$. Since $A \cup B$, $A \cup (e_2 \setminus B)$ and $(e_1 \setminus A) \cup B$ aren't transversals, there exist edges e_4, e_5 and e_6 in $(e_1 \setminus A) \cup (e_2 \setminus B)$, $(e_1 \setminus A) \cup B$ and $A \cup (e_2 \setminus B)$. It follows that $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ does not have a transversal of size 2, because no three of them intersect. This is a contradiction.

Suppose there does not exist an edge e_3 such that $|e_1 \cap e_3| \leq r/2$ and $|e_2 \cap e_3| \leq r/2$. We say the hypergraph is thick. By theorem 3 of [5] the number r is odd and the hypergraph is isomorphic to either K^r_{2r} or two copies of two disjoint copies of $K^r_{(3r-1)/2}$. However these hypergraphs don't satisfy $(6, 2)$. $\square$

Chapter 5

l-Covering Hypergraph

This chapter is about a hypergraph constructed by Bucić, Korándi and Sudakov [4] and how it is used to find a lower bound on $h_r(k,l)$. Let $H = (V,E)$ be an r-uniform hypergraph. We define its l-covering hypergraph to be

$$H_l = (V_l, E_l) = (E, \{\{e \in E \setminus S\} | S \subseteq V, |S| = l\}).$$

Note that when $r = 2$ and $l = |V| - 3$, this is exactly the line graph of H. We now prove the following connection to the (k,l)-covering property.

Lemma 5.1 *Let $K(H)$ be the highest number k such that H satisfies (k,l). Then*

$$\tau(H_l) = K(H) + 1.$$

Proof Consider k elements in $E = V_l$. They are covered by the set S of l elements in V, if and only if $\{e \in E \setminus S\}$ isn't covered by these vertices in V_l. Therefore $\tau(H_l)$ equals the smallest k such that H doesn't satisfy (k,l). □

We can use Lovasz' integrality gap (theorem 2.3) to bound $\tau(H_l) = K(H) + 1$ by the fractional cover number of H_l.

$$\tau^*(H_l) \leq K(H) + 1 \leq (1 + \ln \Delta(H_l))\tau^*(H_l)$$

This is useful, because fractional cover numbers are sometimes easier to work with. When H is complete, we even have an explicit formula for $\tau^*(H_l)$, which we will now prove.

Lemma 5.2 *Let $H = K_n^r$. Then*

$$\tau^*(H) = \frac{n! \cdot (n-r-l)!}{(n-r)!}.$$

Proof First note that

$$|V_l| = \binom{n}{r} \quad \text{and} \quad |E_l| = \binom{n}{l}.$$

Furthermore, H_l is also $\binom{n-l}{r}$-uniform and $\binom{n-r}{l}$-regular. Therefore its fractional cover number is at least $|E_l|/\binom{n-r}{l}$. This bound is attained by assigning a weight of $1/\binom{n-l}{r}$ to each vertex of H_l, because $\binom{n}{l}/\binom{n-l}{r} = \binom{n}{r}/\binom{n-r}{l}$.$\square$

Consequently, the inequality becomes

$$\frac{n! \cdot (n-r-l)!}{(n-r)! \cdot (n-l)!} \leq K(K_n^r) + 1 \leq \left(1 + \ln \binom{n-r}{l}\right) \frac{n! \cdot (n-r-l)!}{(n-r)! \cdot (n-l)!}.$$

Substituting $n = \tau(K_n^r) + r - 1$ and inverting the right inequality gives a new lower bound on τ in terms of k, l and r.

Chapter 6

Bounds for the Case $l = r$

In this chapter we discuss the bounds for the case $l = r$. This case is particularly interesting because of its connection to Ryser's conjecture, see chapter 7.

The following bounds were proven by Bucić, Korándi and Sudakov [4].

Theorem 6.1 *Let $e^r \geq k > r \geq 2$, then*

$$\frac{r^2}{16 \ln k} \leq h_r(k,r) \leq \frac{16r^2 \ln \frac{4r}{\ln k}}{\ln k}$$

The conditions on the parameters can be explained as follows.

- $e^r \geq k$, because $h_r(k,l) = \Theta(r)$ for $k \geq e^r$.
- $k > r$, because any hypergraph satisfies (k,r) for $k \leq r$.
- $r \geq 2$, because $h_1(k,r) = \infty$ whenever $k > r$.

6.1 Lower bound

Note that the maximal transversal number for complete r-uniform (k,l)-hypergraphs is a lower bound on $h_r(k,l)$. Therefore any lower bound on that number will also be a lower bound on $h_r(k,l)$. In chapter 3 we have shown that K_n^r satifies (k,l) for any $n < \frac{rl}{\ln k}$. Therefore

$$h_r(k,r) \geq \left\lceil \frac{r^2}{\ln k} \right\rceil - r.$$

This is an improvement by a constant factor for most of the regime. For values of k greater than $e^{r/2}$, the trivial bound of $h_r(k,r) \geq r$ does best.

6.2 Upper bound for small k

Conversely, any upper bound on $h_r(k,l)$ is an upper bound on the maximal transversal number for complete r-uniform (k,l)-hypergraphs. So the upper bound from Theorem 6.1 will give a sharper upper bound for complete graphs in the case $l = r \geq \ln k$.

6.3 Upper bounds for large k

In the previous section we assumed $k \leq e^r$. For larger k we have an upper bound of

$$\frac{16r^2 \ln \frac{4r}{\ln\lfloor e^r \rfloor}}{\ln\lfloor e^r \rfloor} = O(r).$$

If $k \geq \binom{2r}{r}$, then a different result from [4] says that $h_r(k,r) = r$. Using Stirling's formula, we find that $h_r(k,r) = r$ for all $k \geq 2^{2r-1}/\sqrt{r}$.

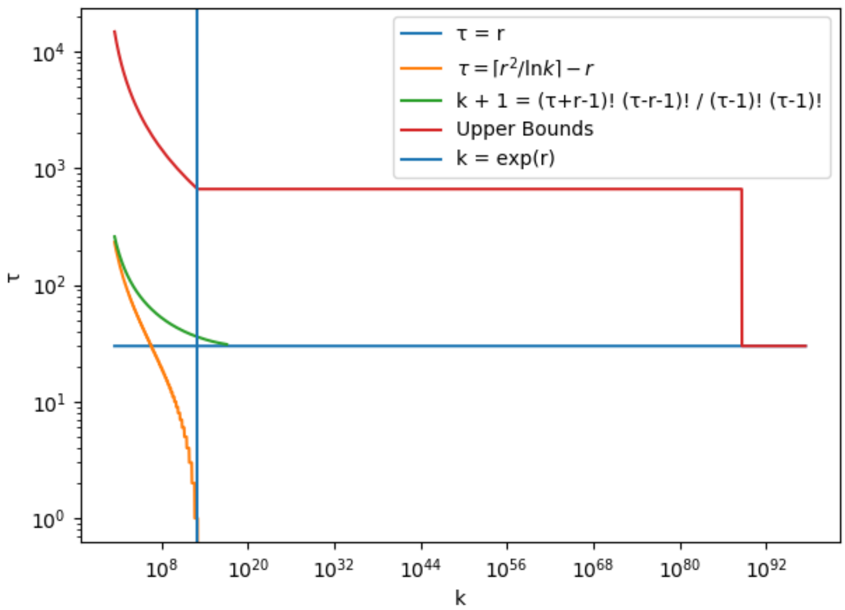

Figure 6.1: Log-log plot of several bounds for the case $l = r = 30$

Chapter 7

Ryser's Conjecture

Recall that the matching number ν of a hypergraph H is the maximum number of pairwise disjoint edges, while the transversal number τ is the size of the smallest set of vertices meeting every edge.

Let H be an r-uniform hypergraph. Since the vertices of any maximal matching cover all edges of H, it holds that $\tau \leq r\nu$.

Note that this bound is tight. Take the complete r-uniform hypergraph on $r\nu + r - 1$ vertices for example. It has matching number ν and transversal number $r\nu$.

It is believed that a stronger bound holds, when the hypergraphs are not only r-uniform, but also r-partite. This is Ryser's conjecture.

Conjecture 7.1 (Ryser) *Let H be an r-uniform r-partite hypergraph with $r \geq 2$. If ν is the maximum number of pairwise disjoint edges in H, and τ is the size of the smallest set of vertices which meets every edge, then $\tau \leq (r-1)\nu$.*

The case $r = 2$ is Kőnig's theorem as discussed in section 2.1. The case $r = 3$ was proven in 2001 by Aharoni using a generalisation of Hall's matching theorem and will be cover in section 7.3. The case $r \geq 4$ remains open.

7.1 Lower bound

We prove the lower bound is tight whenever $r - 1$ is prime by constructing an example of an r-uniform r-partite hypergraph for which $\tau = (r-1)\nu$.

We will use a projective plane. A projective plane Π is a hypergraph (V, E) satisfying the following conditions:

(i) every pair of edges intersect in a unique vertex,

(ii) every pair of vertices is incident to a unique edge,

(iii) there are four vertices such that no three of them are incident to a single edge.

It is a well known fact that there exists a projective plane of size $p^2 + p + 1$ for each prime number p. Let H be such a projective plane minus one point and any edges going through that point. Note that this hypergraph is $(p+1)$-partite and $(p+1)$-uniform. Moreover, it has matching number 1 by property (i) and transversal number p.

Let H be ν disjoint copies of this truncated projective plane. This new hypergraph H is also $p+1$-partite and $p+1$-uniform, but it has matching number ν and transversal number $p\nu$. Therefore equality holds whenever $r-1$ is prime.

7.2 Hall's theorem for hypergraphs

In this section we sketch the proof of the generalisation of Hall's theorem by Aharoni and Haxell [2]. To do so we need some new terminology.

Let $H = (V, E)$ be a hypergraph and let $H' \subseteq E$. We define the *pinning number* $\pi_H(H')$ to be the size of the smallest set of edges $E' \subseteq E$ such that every edge in H' intersects an edge in E' (every edge in H' is "pinned" by an edge in E'). It is similar to the transversal number, but now we are covering edges with other edges instead of with vertices.

The *matching width* $\mathrm{mw}(H)$ of a hypergraph H is the maximal pinning number of its matchings.

$$\mathrm{mw}(H) = \max_{M \text{ matching in } H} \pi_H(M)$$

A *system of disjoint representatives* is a function $f : \mathcal{A} \to \bigcup \mathcal{A}$ such that

$$\forall A, B \in \mathcal{A}, \quad f(A) \in A \text{ and } (A \neq B \implies f(A) \cap f(B) = \emptyset).$$

Theorem 7.2 (Aharoni, Haxell) *Let $\mathcal{A}$ be a family of hypergraphs. If $mw(\bigcup \mathcal{B}) \geq |\mathcal{B}|$ for every $\mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{A}$ has a system of disjoint representatives.*

Sketch of proof The proof is a bit long to explain in detail, so we provide a brief sketch. The main tool used is Sperner's lemma [16].

Lemma 7.3 (Sperner) *Let T be a triangulation of Δ_n, the n-dimensional simplex, and let χ be a colouring of the points of T in $n+1$ colours satisfying the following conditions:*

- *Each vertex of Δ_n is coloured in a different colour.*
- *The points of T on a face of Δ_n are coloured in the colours of the vertices of that face.*

Then there exists a simplex in the triangulation, whose vertices receive all $n+1$ colours.

We associate each hypergraph in $\mathcal{A}$ with a unique colour. The condition on the matching width allows us construct a triangulation T of $\Delta_{|\mathcal{A}|-1}$ satisfying the following conditions.

- The points of T on a face of $\Delta_{|\mathcal{A}|-1}$ are coloured in the colours of the vertices of that face.
- Each point of T is associated with an edge of the corresponding colour.
- Edges corresponding to adjacent points are disjoint.

Applying Sperner's lemma tells us that there are $|\mathcal{A}|$ disjoint and distinctly coloured edges. These form the system of disjoint representatives of $\mathcal{A}$.

To prove Hall's theorem for $G = (V_1 \cup V_2, E)$ we set $\mathcal{A} = \{G[N(v)] : v \in V_1\}$.

7.3 3-Uniform 3-partite hypergraphs

In this section we present Aharoni's proof of the $r = 3$ case of Ryser's conjecture [1].

Theorem 7.4 (Aharoni) *Let H be a 3-uniform 3-partite hypergraph. Then*

$$\tau(H) \leq 2\nu(H).$$

The proof uses a deficiency version of theorem 7.2 and was inspired by a proof of Kőnig's theorem which used a deficiency version of Hall's theorem. So what exactly is deficiency?

The usual Hall's theorem says that bipartite graph $G = (V1 \cup V_2, E)$ contains a complete matching from V_1 to V_2 if and only if $|N(S)| \geq |S|$ for all $S \in V_1$.

The deficiency version says that if $|N(S)| \geq |S| - d$ for all $S \in V_1$, we can still find a partial matching of size $|V_1| - d$. This d is the deficiency number.

In the generalisation, neighbourhoods become hypergraphs and their sizes become matching widths. That way the *deficiency number* of a family of hypergraphs $\mathcal{A}$ becomes

$$\operatorname{def}(A) = \max(0, \max_{\mathcal{B} \subseteq \mathcal{A}} |\mathcal{B}| - \operatorname{mw}(\cup\mathcal{B})).$$

Lastly, the generalisation of the partial matching number is the partial system of representatives, which is just the system of representatives of a subfamily. We will now prove the deficiency version of theorem 7.2

Lemma 7.5 (Aharoni) *Let $\mathcal{A}$ be a family of hypergraphs. Then $\mathcal{A}$ has a partial system of distinct representatives of size at least $|\mathcal{A}| - def(\mathcal{A})$.*

Proof Add the same $\operatorname{def}(\mathcal{A})$ new disjoint edges to each hypergraph in $\mathcal{A}$. This will increase the matching width of each subfamily by exactly $\operatorname{def}(\mathcal{A})$. Now the condition of theorem 7.2 is satisfied, so the new family of hypergraphs has a system of representatives. Removing any new edges from the system of representatives leaves us with a partial system of representatives of size at least $|\mathcal{A}| - \operatorname{def}(\mathcal{A})$. $\square$

We will now use this lemma to prove the main theorem.

Proof (Theorem 8.4) Let $H = (V_1 \cup V_2 \cup V_3, E)$ be a 3-uniform 3-partite hypergraph. Let $\mathcal{A} = \{\{e \setminus \{v\} : v \in e \in E\} : v \in V_1\}$ and note that (partial) systems of representatives corresponds to matchings. If $\operatorname{def}(\mathcal{A}) = 0$, then $\nu(H) = |V_1| \geq \tau(H)$ by lemma 7.5. Otherwise we pick a subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\operatorname{mw}(\bigcup \mathcal{B}) = |\mathcal{B}| - \operatorname{def}(\mathcal{A})$.

It follows from lemma 7.5 that $|\mathcal{A}| - \operatorname{def}(\mathcal{A}) \leq \nu(H)$. Moreover, note that $\tau(\bigcup \mathcal{B}) \leq 2\operatorname{mw}(\bigcup \mathcal{B})$, since each pinning edge covers two vertices, and $\tau(\bigcup \mathcal{A} \setminus \bigcup \mathcal{B}) \leq |\mathcal{A}| - |\mathcal{B}|$, because we can cover using the first vertex class. Combining these three equations gives

$$\begin{aligned}
\tau(H) &\leq \tau\left(\bigcup \mathcal{A}\right) \leq \tau\left(\bigcup \mathcal{B}\right) + \tau\left(\bigcup \mathcal{A} \setminus \bigcup \mathcal{B}\right) \\
&\leq 2\operatorname{mw}\left(\bigcup \mathcal{B}\right) + |\mathcal{A}| - |\mathcal{B}| \leq |\mathcal{A}| + |\mathcal{B}| - 2\operatorname{def}(\mathcal{A}) \leq 2\nu(H),
\end{aligned}$$

which concludes the proof. $\square$

7.4 Connection to the main problem

Ryser's conjecture is equivalent to

$$h_r^r(l+1, l) \leq (r-1)l,$$

where $h_r^r(k, l)$ is defined as the maximal transversal number any r-uniform r-partite hypergraph satisfying the (k, l) property can have.

Proof Let H be an r-uniform r-partite hypergraph with matching number l. Then the right-hand side equals $(r-1)\nu(H)$. Since H does not have $l+1$ independent edges, it satisfies $(l+1, l)$. Therefore $\tau(H) \leq h_r^r(l+1, l)$. So the inequality implies Ryser's conjecture.

On the other hand, since $h_r^r(l+1, l)$ is bounded, there exists a r-uniform r-partite hypergraph H' with matching number at most l and transversal number $h_r^r(l+1, l)$. Thus Ryser's conjecture implies

$$h_r^r(l+1, l) = \tau(H') \leq (r-1)\nu(H') \leq (r-1)l. \qquad \square$$

Note that $h_r^r(k,l)$ is bounded above by $h_r(k,l)$. In particular $h_r^r(l+1,l) \leq lr$ by proposition 4.2. Conversely any constructions for lower bounds on $h_r^r(l+1,l)$ to prove Ryser's conjecture give lower bounds on $h_r(l+1,l)$ as well.

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