Ramsey numbers of hypercubes

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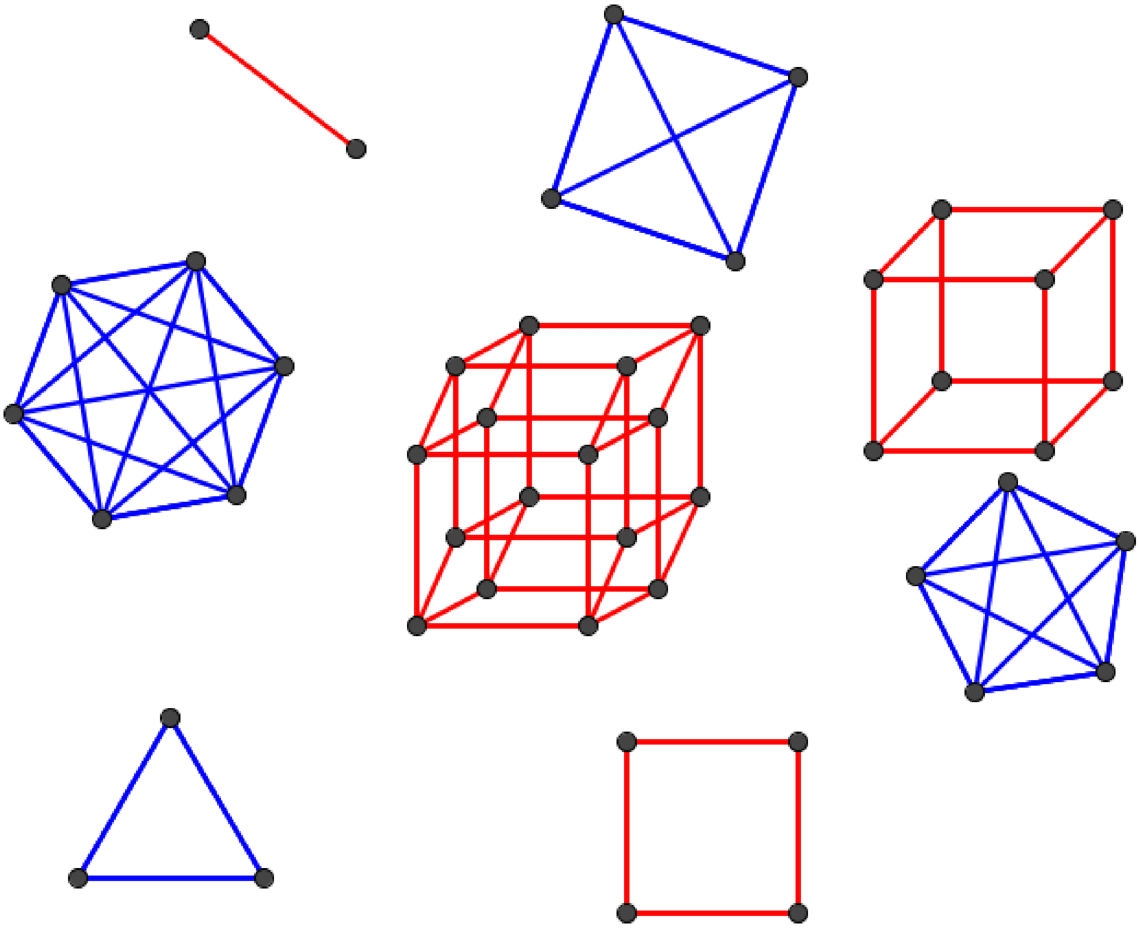
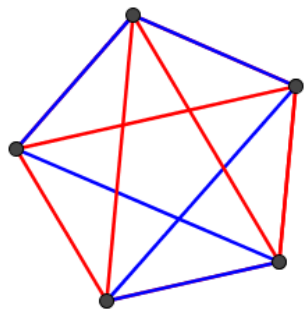
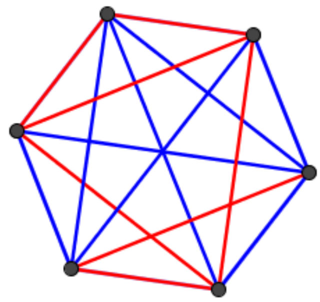


1 Ramsey Theory

In this essay we shall investigate two ground-breaking theorems in Ramsey theory. Ramsey theory is a branch of graph theory looking for order in disorder. A graph is a set of points, the vertices, joined by some lines, the edges. In this essay we will only consider graphs with a finite number of vertices. If there is an edge between each pair of vertices, we say the graph is complete. We denote the complete graph on n vertices with K_n.
Now suppose we colour each edge of K_n either red or blue. Can we find patterns in this? Can we find a blue triangle? What about a red triangle? It turns out that if you have six vertices you can always find one of them.





As you can see K_5 need not contain a red or blue triangle. For K_6 we failed to find such an arrangement. Can you see why?

Instead of searching for red or blue triangles, we could consider a different pair of graphs, one completely red and one completely blue. Can we always find one of the two, if we search a sufficiently large complete graph? Ramsey's theorem says we can.

Ramsey's Theorem. *Consider a blue graph G_1 and a red graph G_2, then there exists N such that every red-blue-colouring of a complete graph of size N contains one of the two. The minimal such N called is the Ramsey number of G_1 and G_2.*

Proof. We will prove this theorem by induction on the size of the graphs, i.e. the number of vertices they contain.

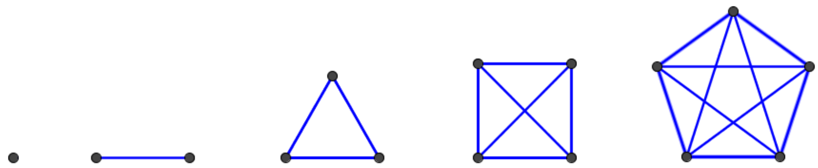
Induction basis: If $|G_1| = |G_2| = 1$, the complete graph on one vertex already suffices.

Induction hypothesis: Any red-blue-colouring of K_N contains either a blue G_1 or a red G_2 for all G_1, G_2 with $|G_1| + |G_2| < n$.

Induction step: Let $|G_1| + |G_2| = n$.
Consider red-blue-coloured K_{2N} and pick one vertex v. This vertex has $2N - 1$ edges coming out of it. By the pigeonhole principle at least N of them are red or at least N of them are blue.
We will first consider the case where N of the edges are red. The vertices they lead to form a graph K'. Let G_2' be G_2 minus one of its vertices. Since $|G_1| + |G_2'| = n - 1$, there must be a blue G_1 or a red G_2' in K'. A blue G_1 clearly finishes our proof. However a red G_2' also does, as v can serve as the vertex deleted from G_2.
The proof of the case where N of the edges are blue is very similar. $\square$

2 Cliques and hypercubes

The most famous Ramsey numbers are those of cliques. Cliques are probably the most fundamental subgraphs as they are the analogue of the complete graph. When we were looking for red or blue triangles, we were looking for monochromatic cliques of order 3 (cliques on 3 vertices whose edges all have the same colour).

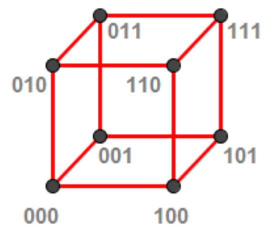


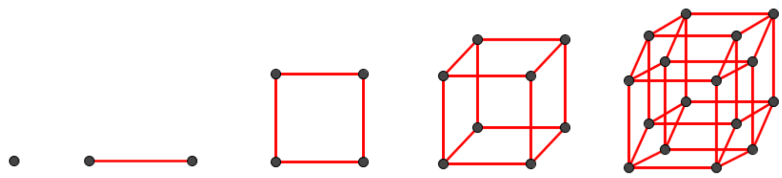
Here are the five smallest examples of blue cliques. Can you predict how the sequence continues?

There has been a lot of research to the Ramsey numbers of two cliques. However they turn out to be extremely hard to find and very few Ramsey numbers of cliques are known exactly. The difficulty of the problem is illustrated by the following quote.

> Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack. - Paul Erdős

There are plenty of other subgraphs of interest besides cliques. However there is one particularly important: the hypercube. Hypercubes play a key role as they are the graphical interpretation of powersets. The n-dimensional hypercube has its vertices labelled by elements of $\{0,1\}^n$ with edges joining them if and only if their labels only differ in one coordinate.





Notice we obtain subcubes by keeping some coordinates fixed. This makes hypercubes very special.

Cliques and hypercubes are immensely different and yet equally fundamental. It would be great to know which graphs contain them. This brings us to the following question.

Problem. *What is the value of $R(Q_n, K_s)$, the Ramsey number of the red cube of dimension n and the blue clique on s vertices?*

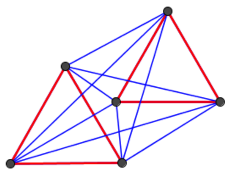
The problem appears to be as intractable as its all-clique equivalent, but let's give it a try

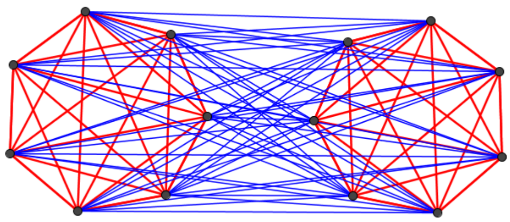
3 The lower bound

To find out more about this number, there are two things to look for, an upper bound and a lower bound. Ideally the two are equal. In this case, finding an upper bound N means taking into account all possible colourings of K_N and proving they all contain a copy of a blue clique K_s or a red hypercube Q_n. For a lower bound N' it is sufficient to construct a red-blue-colouring of $K_{N'}$ with neither the blue clique nor the red hypercube.

Constructing a lower bound seems easier, because we just need one example. However success is not guaranteed. The example we need could be incredibly messy. If we want our example to work as n and s go to infinity, it better be something simple.

Let's start with the case $s = 3$. We do not want any blue triangles in the graph. An easy way to achieve this is to use a bipartite graph (a graph with no odd cycles and hence no triangles). We take two red cliques and join them with blue edges. The blue edges form a bipartite graph. Now we don't want the red cliques to contain a red hypercube of dimension n, so we give them $2^n - 1$ vertices each. Using this construction we find $R(Q_n, K_3) \geq (2^n - 1) + (2^n - 1) + 1 = 2^{n+1} - 1$.

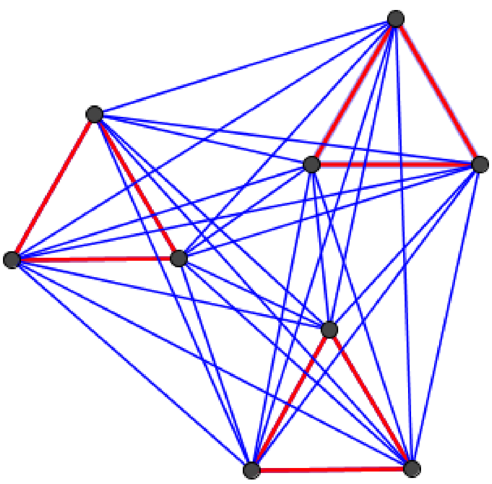




The first graph contains no red square, the second graph contains no red cube. Neither contains a blue triangle.

These examples show that $R(Q_2, K_3) > 6$ and $R(Q_3, K_3) > 14$.

Now we can extend this idea to find a bound for general cliques K_s. Take $s - 1$ red cliques, each on $2^n - 1$ vertices, connected by blue lines. Red hypercubes must lie within the same red clique and are bounded by $2^n - 1$. Blue cliques cannot have two points in the same red clique and are bounded by $s - 1$. This gives us a lower bound of $(2^n - 1)(s - 1) + 1$.



The example contains no red square and no blue clique of order 4. This shows that $R(Q_2, K_4) > 9$.

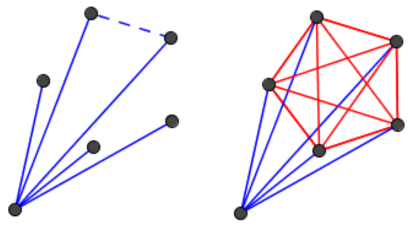
Are these bounds tight? No idea, therefore we need an upper bound!

4 The upper bound: finding the exponent

We know an upper bound exists. By Ramsey's Theorem sufficiently large graphs always contain a blue K_s or a red Q_n. Changing our inductive proof of Ramsey's theorem, we can prove there is an upper bound of $\binom{2^n+s-2}{s-1}$. For $s = 3$ this is $2^{n-1}(2^n + 1)$. Now we know the growth is exponential with a growth rate between the 2 and the 4. However we can do much better.

4.1 Greedy algorithm for cubes

Let us first consider the case $s = 3$. This case has a useful property: the absence of blue triangles. This might sound trivial, but consider the following. Take a vertex, v say, and consider $N_B(v)$ the set of its blue neighbours. No x and y in $N_B(v)$ can have a blue edge between them. Otherwise v, x and y form a blue triangle. This means $N_B(v)$ is a red clique.



So, how does the algorithm work? We take a hypercube and want to assign a vertex of the graph to each of its vertices (which we from now on will call corners to avoid confusion). Every corner of the hypercube has n neighbours. If there is already a vertex assigned to on one of these neighbours, that limits our possibilities. There can be at most $2^n - 1$ blue edges coming out of these vertices. So $n(2^n - 1)$ vertices are "blocked". We have already assigned at most $2^n - 1$ vertices as well. So if we start out with $(n+1)(2^n - 1) + 1$ vertices, then there must always be one left.

For general s, the blue neighbourhood of a point can not contain a blue $(s - 1)$-cliques, otherwise the point and its neighbours form an s-clique. We can generalize this to greater cliques using induction.

Induction hypothesis: Suppose that the two-coloured complete graph on $\frac{(n^{s-1}-1)}{n-1}(2^n - 1) + 1$ vertices always contains a red hypercube of dimension n or a blue clique of size s.

Induction step: Take a two-coloured complete graph on $\frac{(n^s-1)}{n-1}(2^n - 1) + 1$ vertices. Again we take a hypercube and try to assign a vertex of the graph to some corner v. So far we have assigned vertices to at most $2^n - 1$ corners. If we cannot assign any more vertices, this means that each of the $\frac{(n^s-n)}{n-1}(2^n - 1) + 1$ vertices left is a blue neighbour of a vertex assigned to one of the n neighbours of v. Therefore one of these vertices has $\frac{(n^{s-1}-1)}{n-1}(2^n - 1) + 1$ blue neighbours. By the induction hypothesis, this neighbourhood contains either a red Q_n or a blue K_s. Since a blue K_s in a blue neighbourhood implies a blue K_{s+1}, we are done.

This is cool as it tells us the growth rate is 2. Unfortunately there is still a polynomial factor of $n^{s-2} + n^{s-3} + ... + n$.

5 The upper bound: up to a factor

We have a lower bound that goes like 2^n. Is there an upper bound that grows at the same speed. That turns out to be the case. This discovery was made in 2012 by Conlon, Fox, Lee and Sudakov. The number was found up to some constant factor. For simplicity we will just consider the case $s = 3$, i.e. a blue triangle.

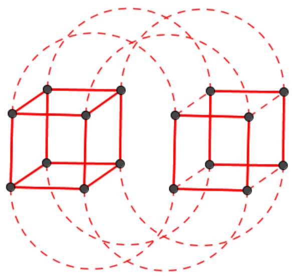
Theorem 1 (Conlon, Fox, Lee, Sudakov)**.** *Consider a colouring of the complete graph K_N on the vertex set $[N] = \{1, 2, ..., N\}$ for $N \geq 7000 \cdot 2^n$ with two colours, red and blue, and assume that there are no blue triangles. Then this colouring contains a red Q_n.*

Proof. Just as we want a construction for the lower bound, we want a construction for the upper bound. We want a proof that in each and every triangleless mess, we can find a hypercube.

How do you find a structure as complex as a hypercube? That is hard, if not impossible. However we can build smaller cubes and try to assemble a bigger one out of them. A hypercube of dimension n is like two hypercubes of dimension $n - 1$ stuck together.

Our strategy is to take red hypercubes and assemble them into a bigger hypercube. The things we need to proof are that we have enough blocks (the hypercubes) and enough glue (red edges) to complete our building (the big hypercube). The idea of the proof for our sufficiently large n is as follows:

1. Algorithm to pick red cliques from K_N.
2. Algorithm to assign them to subcubes of Q_n.
3. Proof that Q_n is now fully covered with cliques.
4. Proof that we can orient the cliques to have exclusively red edges connecting them.



5.1 Picking red cliques

Our problem is that we do not know how the cubes will fit together. We want to have freedom to rearrange the points at the glueing stage. To do so we need to find red cliques (instead of cubes) of order 2^k. This is going to restrict our choice initially, but there seems to be no alternative.
So let us pick red cliques of order a power of two. We pick them in decreasing order of size, add them to our set S and remove them from the vertex set. If we have not yet used half of the vertices, then add the set of the remaining vertices to S as well and declare the codimension to be zero (we will come to that later).

5.2 Assignment to special subcubes

How subcubes fit together is complicated. They can be arranged in different ways. That is why we will introduce the concept of special subcubes. The advantage of using special subcubes is, that we can easily describe when two subcubes are disjoint or adjacent.

Definition (Special subcube)**.** A special subcube of Q_n of codimension d is a a set

$$\{a_1, a_2, ..., a_d, x_{d+1}, ..., x_n \ : x_i \in \{0, 1\}\},$$

where $a_1, ..., a_d \in \{0, 1\}$.

They are disjoint if they differ in at least one coordinate and adjacent if they differ in exactly one coordinate.

Now we will start assigning our cubes to initial subcubes using the following algorithm

We assign as large a cube as possible, but no larger than the largest assigned cube, such that the following conditions hold:

- The special cube has a quarter of the size of the subgraph.
- If C and C' are two assigned cubes. There may be at most $|C||C'|/16\delta^2$ edges between them, were δ is the maximum of the codimensions of C and C'

We will continue to assign cubes as long as possible. Then there are two scenarios. Either we are finished or we are not finished.

5.3 Covered cube

Of course we want to be in the first scenario. Are we in there? Yes, in the following five steps we will prove that assigning cliques to special subcubes as described above will leave no vertex of the hypercube uncovered.

1. There are at least $N/2 - 4 \cdot 2^n = 7000 \cdot 2^n/2 - 4 \cdot 2^n = 3496 \cdot 2^n$ vertices left.
   Suppose at most $32i^3$ sets of codimension i.

   $$\sum_{i=0}^{\infty} 32i^2 \cdot 4 \cdot 2^{n-i} = 128 \cdot 2^n \cdot \sum_{i=1}^{n} \frac{i^3}{2^i} = 3328 \cdot 2^n$$

   This is a contradiction. So there exists an i such that there are at more than $32i^3$ sets of codimension i.

2. Pick an uncovered vertex and consider its initial subcube of codimension i.

3. It has at most i neighbouring cubes with a set assigned to it.

4. The number of blue edges is at most 2^{n-i+3}. Between sets we want at most $\frac{1}{16i^2}|S_C||S_{C'}| = 2^{n-i+3} \cdot |S_C|/32i^2$ blue edges. So for each neighbouring cube there are at most $32i^2$ bad sets of codimension i.

5. We can pick a good set of codimension i.

Now the cube has been covered by sets, we must prove that this assignment indeed induces a red hypercube.

5.4 Completely red

To find a red hypercube it would be helpful to reduce the blue degree of our sets. We can do so by giving up on some vertices and reducing the set size. First we order our sets from small to large $S_1, ..., S_l$. For each $i < j$ we remove all vertices from S_i with more than $|S_j|/8\delta$ blue neighbours in S_j. There are at most $|S_i||S_j|/16\delta^2$ blue edges between the sets. So we remove at most $|S_i|/2\delta$ vertices. Since we do this at most δ times (the maximum number of neighbouring cubes that are greater), we remove at most half the vertices. The set T_i of vertices left has size at least $2 \cdot 2^{n-\delta}$. The vertices of T_i have at most $|T_j|/4\delta$ neighbours in T_j.

Now we have a stronger condition on the degree, we can start embedding the red hypercube. There are at most $|T_i|/4$ vertices we cannot pick because of the neighbouring cubes. There are at most $|T_i|/2$ neighbours already embedded. Furthermore it has $n - \delta$ neighbours within its subcube, which can forbid $2^{n-\delta-1}/n$ vertices each. This adds up to less than $|T_i|$ vertices, so we can always find one. $\square$

This proof can be extended to general s. We won't do that here, as we are on a quest to an even more precise answer.

6 The upper bound: exactly

We've reduced our error margin to a factor. That is more than we could hope for. So are we done? Actually we can proof it exactly. For sufficiently large n the lower bound seems to be an upper bound as well.

Theorem 2 (Fiz Pontiveros, Griffiths, Morris, Saxton, Skokan)**.** *For all s there exists n_0 such that*

$$r(K_s, Q_n) = (s-1)(2^n - 1) + 1$$

for all $n \geq n_0$.

Proof. Now we will explain the proof this theorem for general s. We want the lower and the upper bound to be exactly the same. So we want our lower bound configuration to be optimal. Therefore we attempt the following strategy. We prove that our graph is either very similar to our lower bound example (but bigger and hence containing a red cube) or very differently structured (and hence containing a red cube).

The proof will consist of the following steps:

1. Partition almost all vertices into equally sized internally dense red sets.
2. Embed the cube, when the graph is very similar:
   the vertex set is partitioned into $s-1$ red cliques.
3. Embed the cube, when the graph is very different:
   the sets are joined by small, disjoint, red complete bipartite graphs.
4. Prove that we must be in one of the cases.

We will discuss steps 1, 2 and 3.

6.1 Partitioning the graph

We want to divide a high fraction of the vertices into sets which have very few internal blue edges. Let's say we would like the sets to cover at least $(1-\epsilon)N$ and the blue degree to be at most $\epsilon/8$ of the set size. Ideally the sets are similarly sized as in the lower bound example.

We set $a_0, ..., a_K$ to be a sequence of integers with $a_0 = N$ and $a_{i+1} \leq \frac{\epsilon}{8} \cdot a_i$ for every $0 \leq i \leq K-1$. We hope that if we choose K (and hence N) high enough, we will be able to find a family $\mathcal{U}$ of disjoint set of vertices such that

- The sets cover $(1-\epsilon)N$ of the vertices.
- The sets all have size a_i.
- The maximal blue degree of the sets is less than a_{i+1}.

Induction is always a good place to start your search for a proof. Therefore we make a definition of a more general case.

Definition. We say a set W of vertices is (β, I)-good, if for all $i < j$ with i and j both in the interval I, it contains a family of disjoint subsets $\{U_1, ..., U_n\}$ such that

- The subsets together contain at least $\beta \cdot |W|$ of the vertices.
- Every subset has size a_i.
- The maximal blue degree within the subsets is at most a_j.

Note that we have to proof that V is $(1-\epsilon, [i, i+1])$-good for some i. Our strategy is as follows. We look for some disjoint subsets $U_2, ..., U_s$ and a series of intervals $I_2 \supset ... \supset I_s$ of size at least 2 such that

- $|U_2| + ... + |U_s| \geq (1-\epsilon/2)N$
- U_r is $(1-\epsilon/2, I_r)$-good for every $2 \leq r \leq s$

Since each U_r is $(1-\epsilon/2, I_s)$-good, we know W is $((1-\epsilon)^2, I_s)$-good and hence $(1-\epsilon, I_s)$-good. One can prove by induction on s that such sets exists. However we will omit the proof.

6.2 Similar

First we consider the case where the graph looks similar to the lower bound example. The graph is partitioned into $s-1$ red cliques $S_1, S_2, ..., S_{s-1}$ and a set S_0 of the leftover vertices. Without loss of generality we may assume that S_1 is the greatest of the cliques. Therefore $|S_0| + |S_1| \geq |S_0| + \frac{(s-1)(2^n-1)+1-|S_0|}{s-1} > 2^n - 1$. Since the sets contain at least 2^n vertices, they have enough vertices to contain a red hypercube. If $|S_1| \geq 2^n$, we can just pick 2^n random vertices from the set and they will form a red hypercube. If S_1 contains less than 2^nvertices, then we pick $2^n - |S_1|$ vertices from S_0 and call this set Y.

We will prove that under the following conditions the graph on $S_0 \cup Y$ does indeed contain a red hypercube.

- $|S_1| \geq \frac{7}{8} \cdot 2^n$.
- Every vertex in S_0 has at most $|S_1|/n^2$ blue neighbours in S_1.

Each vertex of Q_n has a coordinate in $\{0, 1\}^n$. We define the l^{th} layer as the set of vertices with l ones and $n - l$ zeroes. We want that vertices which are adjacent in the cube, are assigned to vertices with a red edge between them in the graph. Vertices that are adjacent in the cube differ in exactly one coordinate and therefore their layer numbers also differ by exactly 1.

Now we start assigning vertices in Y to the vertices in the $0^{\text{th}}, 2^{\text{nd}}, 4^{\text{th}}, 6^{\text{th}}, ...$ layers until we run out of vertices. Because $|Y| \leq 2^n/8$, we have not yet reached the top half of the layers. There are no problems so far, because the vertices of the cube aren't adjacent and the colours of the edges between them does not matter.

Now we start assigning vertices of X to the vertices in the $1^{\text{st}}, 3^{\text{rd}}, 5^{\text{th}}, 7^{\text{th}}, ...$ layers until we have covered all vertices of the cube. While doing so we pay attention that we do not create blue edges between adjacent vertices in the cube. Is that possible?

Each vertex of the cube has at most n neighbours that already have a vertex of Y assigned to it. Each vertex of Y has at most $|S_1|/n^2$ blue neighbours. This forbids at most $|S_1|/n$ vertices. So we might have a problem when we have only $|S_1|/n$ vertices left. However we have then already started filling high layers that are far from the vertices assigned to Y and no vertex is forbidden.

6.3 Different

Now we consider a different kind of graph. This graph has a subgraph with very few internal blue edges. It is summarised in the following statement.

Given any $\epsilon > 0$ and $k \in \mathbb{N}$ there exists n_0 such that for every $n \geq n_0$, $s \leq \log_{k+1}(n)$ and H a two-coloured complete graph on $(1+\epsilon)2^n$ vertices with no blue K_s and

$$d_B(u) \leq \frac{2^n}{\log_{(k)}(n)}$$

for every $u \in V(H)$, then $Q_n \subset H_R$.

The proof goes as follows.

1. Define a particular assignment of hypercubes to subsets of the graph.
2. Use induction to assign hypercubes of dimension $d(r)$ to subsets of the graph for all $0 \leq r \leq k+1$.
3. Show that an assignment of a hypercube of dimension $d(k+1)$ enables us to embed a red hypercube.

6.3.1 Definitions

First we define the dimension function used above

$$d(r) = \begin{cases} (\log_{(k+2-r)} n)^3 & \text{if } r \in \{1, ..., k+2\}, \\ 0 & \text{otherwise} \end{cases}.$$

As we are going to work with initial subcubes, we define for $x, y \in Q_{d(r)}$

$$t(x,y) \begin{cases} \min\{p : x_p \neq y_p\} & \text{if } x \neq y, \\ r+1 & \text{otherwise} \end{cases}.$$

Now for all $0 \leq r \leq k+1$ we define a level-r assignment $A : Q_{d(r)} \to \mathcal{P}(H)$ as a fuction from points on the subcube to disjoint subsets of H such that

- For all $x \in Q_{d(r)}$ we have $|A(x)| = \left(1 + \frac{k+2-r}{k+2}\epsilon\right) 2^{n-d(r)}$.
- For all $x, y \in Q_{d(r)}$ which are equal or adjacent, and all $v \in A(y)$ we have

$$|N_B(v) \cap A(x)| \leq \frac{2^{n-d(r)}}{d(t(x,y))^{4(k+2-r)}}.$$

6.3.2 Induction

Using induction on r one can prove that a level-$(k+1)$ assignment exists. For the level-0 assignment we can simply assign the only vertex to the whole of V. We go through all the details of the induction step, but we will prove the key lemma used.

Lemma 1. *For all $d \geq 0$ and $X \in V(G)$ there exists $Y \subset X$ such that*

- $|Y| \geq 2^{-(s-2)d}|X|$ *and*
- $|N_B(v) \cap Y| \leq 2^{-d}|Y|$ *for all $v \in Y$*

Proof. We will prove this lemma by induction on s.

Induction basis: $s = 3$
Suppose no vertex has $2^{-d}|X|$ blue edges. Then we can pick Y to be all of X.
Suppose a vertex has that many blue edges. Than its blue neighbours form a red clique of size at least $2^{-d}|X|$ and we can pick Y to be this clique.

Induction hypothesis: We can find Y in case and $s = k$.

Induction step: We would like to prove the case $s = k+1$. Therefore we have to find a set of size Y of size at least $2^{-(k-1)d}|X|$ such that the blue degree of any vertex within Y is at most $2^{-d}|Y|$.
Suppose no vertex has $2^{-d}|X|$ blue edges. Then we can pick Y to be all of X.
Suppose a vertex has that many blue edges. Than its neighbours form a set Z of size at least $2^{-d}|X|$ that does not contain an k-clique (as together with the vertex it would be a $k+1$-clique). Now we can use the induction hypothesis to find Y in Z such that $|Y| \geq 2^{-(k-2)d}|Z|$ and $N_B(v) \cap Y| \leq 2^{-d}|Y|$. Note that $|Y| \geq 2^{-(k-2)d}|Z| \geq 2^{-(k-2)d} \cdot 2^{-d}|X| = 2^{-(k-1)d}|X|$.
So we are done! □

Using a probabilistic argument we you can find a Y of any size between $\log|X|2^{d+3}$ and $2^{-(s-2)d}|X|$ such that

$$|N_B(v) \cap Y| \leq 2^{-d+1}|Y| \text{ for all } v \in Y$$

6.3.3 Embedding

Now we have found a level-$(k+1)$ assignment A of Q_n to H, we would like to find an embedding ϕ of Q_n into the graph. We want to use our assignments, so we send every corner of Q_n to a vertex in the image of the corresponding initial subcube of codimension $d(k+1)$. We do this corner by corner. The level-$(k+1)$ assignments make sure we never run out of vertices.

□

7 Conclusion

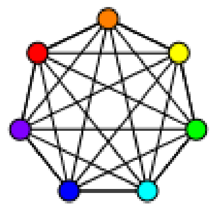
The Ramsey number of the clique has been found. So, what's next? There are still many questions left concerning the Ramsey number of the hypercube. For example, we know that our formula works for sufficiently large n. But what is sufficiently large? And how does the number behave for lower n? Another question to ask is what happens if we replace the clique by some different graph H? Well that question is answered.

Theorem 3 (Fiz Pontiveros, Griffiths, Morris, Saxton, Skokan)**.** *Let H be a graph. Then there exist an n_0 such that*

$$r(H, Q_n) = (\chi(H) - 1)(2^n - 1) + \sigma(H)$$

for every $n \geq n_0$.

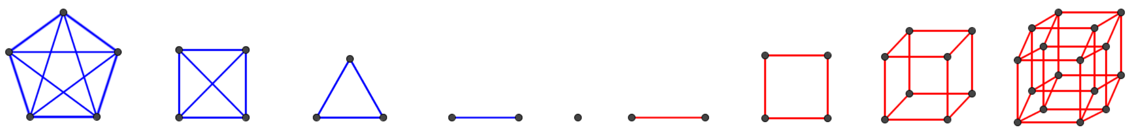
Here $\chi(H)$ is the chromatic number, the minimum number of colours you need to colour the vertices of a graph such that no two adjacent vertices have the same colour. In case of K_s this is s, as all vertices are neighbours.
The other number $\sigma(H)$ denotes the smallest possible colour class in such a vertex-colouring of H using $\chi(H)$ colours. In case of a clique this number is 1, as there is exactly one point in every colour.



The proof turns out to be very similar to that of the clique. For the lower bound, we can just take $\chi(H) - 1$ red cliques of size $2^n - 1$ and one red clique of size $\sigma(H) - 1$. We connect the cliques by blue edges. None of the cliques is big enough to contain a red hypercube (if we pick $n \geq \log \sigma(H)$). Now we give the vertices in each clique a different colour. This gives a valid vertex-colouring of the graph in at most $\chi(H)$ colours. One of the colours does only occur $\sigma(H) - 1$ times. Therefore the graph cannot contain a blue H. The proof for the upper bound is more complicated, so we leave it out.

The theory of Ramsey asks simple questions about complex structures. This often leads to complicated answers. The Ramsey number of the hypercube is one of the exceptions to the rule. We can find a formula for a Ramsey number, something thought to be impossible. In 2012 there was a breakthrough and an upper bound was found, just a constant factor from the lower bound. More results followed and in 2013 the proof was completed.

Will more order be found in the chaos? Will we ever know $R(K_6, K_6)$? Maybe.



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